Long-Term Bond Yields:
A No-Arbitrage Perspective

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Decomposition of bond yields

It comes from the condition for the **absence of arbitrage opportunities**

... which is the **risk–return relation** that applies to all assets

- It does not depend on a specific model
- It only says "you can't get something for nothing"

The decomposition provides

- Structure
  - Interpreting the way the yield curve moved in the past
  - Thinking about how the yield curve might behave in the future
    ("stress-testing" scenarios based on what the FOMC might say or do)
- Constraint
  - Movements in the term structure that cannot be reasonably described in terms of the components of the framework — are not themselves reasonable
- **A different emphasis and/or perspective**
  - Focuses on the **policy path**, which is under the control of the FOMC
  - Brings the way in which the FOMC operates and communicates into direct focus
    Connection between amount of risk (in long-term bonds) and uncertainty about next meeting's policy
  - Plays down the **direct** role of inflationary expectations and real rates (still there if desired)
  - Plays down the role for supply and demand (except via specialness and the repo market)

The decomposition (for default-free bonds)

$$\text{bond yield} = \overbrace{\text{expected path of short-term rate}}^{(1)} + \underbrace{\text{risk premium}}_{(2)} - \underbrace{\text{convexity}}_{(3)} - \underbrace{\text{specialness}}_{(4)} \qquad (\dagger)$$

(1) By itself, the expected path of short-term interest rates delivers the **Expectations Hypothesis**
  - People used to claim the EH was a no-arbitrage condition — it's not

(2) The risk premium depends on the **price of risk** and the **amount of risk**
  - **Price of risk** is the same for all bonds (all assets, in fact)
  - **Amount of risk** is bond-specific (asset specific)
    - Amount of risk increases with term to maturity
    - Explains why the yield curve slopes upward on average

(3) Convexity (related to risk)
  - More bond-specific risk → more convexity → lower yield
    - Convexity increases with term to maturity
    - The effect can be important for longer-term maturities (it's unimportant for short-term maturities)
    - Explains why the STRIPS curve can slope **downward** beyond 20 years

(4) Specialness
  - Specialness in the repo market reflects demands for one security versus another
  - All of the supply and demand stuff flows through here
  - Explains why on-the-run Treasury securities often have lower yields

The **no-arbitrage condition** for an asset

$$\underbrace{\frac{E_t[P_{t+1}] - P_t}{P_t} + \frac{D_t}{P_t}}_{\text{expected return}} = R_t + \mathtt{RP}_t \qquad (\dagger)$$

- P_t is the **price** of the asset at time t
  Realized capital gain: $P_{t+1} - P_t$
- D_t is the **dividend** paid at time t
- R_t is the one-period **risk-free interest rate**
- $\mathtt{RP}_t$ is a **risk premium** for this asset

Note: (†) follows from the "martingale" property of deflated asset prices

- "If there's a possible gain, then there's a possible loss too"

1. Details regarding the **risk premium**

$$\mathtt{RP}_t = (\text{price of risk})_t \times (\text{amount of risk})_t$$

- (price of risk)$_t$ is the same for all assets
- (amount of risk)$_t$ is specific to this asset
  - risk is associated with capital gains
- More generally, the risk premium is a sum over different "types" of risk:

$$\mathtt{RP}_t = \sum_i (\text{price of risk})_{it} \times (\text{amount of risk})_{it}$$

Can be understood as a premium earned for **covariance** risk

The no-arbitrage condition: Expressed in growth-rate terms

continuously compounded

1. Rearrange, take logs, and approximate
2. The **no-arbitrage condition**

$$\underbrace{E_t[\log(P_{t+1}/P_t)]}_{\text{expected growth rate}} + d_t = R_t + \underbrace{\mathtt{RP}_t - C_t}_{\text{risk-related}} \qquad \text{(no-arb)}$$

- R_t is the one-period risk-free interest rate
- $\mathtt{RP}_t$ is a risk premium
- **Dividend rate**: $d_t = \dfrac{D_t}{P_t}$
- **Jensen's inequality** (the source of "convexity")

$$\log(E_t[P_{t+1}/P_t]) \approx E_t[\log(P_{t+1}/P_t)] + C_t$$

where

$$C_t = \frac{1}{2}\,\mathrm{Var}_t[\log(P_{t+1}/P_t)]$$

  - It's exact when $\log(P_{t+1}/P_t)$ is normally distributed

Note: The expected growth rate can be negative even when the expected return is positive if the variance is large enough!

3. We will use (no-arb) to build up long-term bond yields

Default-free zero-coupon bonds: Prices, yields, and forward rates

A zero-coupon bond makes a single payment at a fixed time in the future

1. Prices
   - P_t^n is the **price of an n-period bond** at time t
   - P_{t+1}^{n-1} is the price of this bond at $t+1$ (it becomes an $(n-1)$-period bond)
   - $P_{t+n}^0 = 1$ is the price of this bond at maturity
2. Yield-to-maturity
   - Y_t^n is the yield-to-maturity for an n-period bond at time t

$$Y_t^n = \frac{\log(1/P_t^n)}{n}$$

3. Forward rates
   - f_t^n is the forward rate (from $t+n-1$ to $t+n$) at time t

$$f_t^n = \log(P_t^{n-1}/P_t^n) = n\,Y_t^n - (n-1)\,Y_t^{n-1}$$

   - The yield on an n-period bond is the average of the forward rates

$$Y_t^n = \frac{1}{n}\sum_{s=1}^{n} f_t^s$$

The idea: Forward rate is the rate for a one-period loan in the future

- No cash flow at time t: **Buy** 1 n-period bond for P_t^n and **Sell** an equal value of $(n-1)$-period bonds (i.e., sell P_t^n/P_t^{n-1} bonds)
- At time $t+n-1$: **Pay** P_t^n/P_t^{n-1} on the bonds sold
- At time $t+n$: **Receive** 1 for the bond purchased

The one-period yield is $\log(P_t^{n-1}/P_t^n) = f_t^n$

Long-term default-free bonds: "Dividends"

1. Dividend rate

$$d_t = \frac{D_t}{P_t^n} = \frac{\texttt{Coupon payment} + \texttt{Repo earnings}}{P_t^n} \approx \texttt{CR} + S_t$$

   - Coupon rate: $\texttt{CR}$
     - For zero-coupon bond $\texttt{CR} = 0$
     - For coupon bond: $P_{t+n}^0 = 1 + \texttt{CR}$ and therefore $\log(P_{t+n}^0) \approx \texttt{CR}$
   - **Repo spread**: S_t
     - The spread between the **general collateral rate** and the **specific collateral rate** in the repo market (i.e., repurchase agreement)
     - If a bond is on special then $S_t > 0$ (otherwise $S_t = 0$)
     - Repo earnings equal $S_t \times P_t^n$

2. How to **capture the repo earnings** (if you possess a bond that's on special)
   - Use the bond as collateral to **borrow at the specific collateral rate**
   - **Lend** the funds you borrowed **at the general collateral rate**
   - Earn the difference between the borrowing and lending rates (the spread)

3. The no-arbitrage condition (no-arb) becomes

$$\underbrace{E_t[\log(P_{t+1}^{n-1}/P_t^n)]}_{\text{expected growth rate}} + \underbrace{\texttt{CR} + S_t}_{\text{dividend rate}} = R_t + \underbrace{\texttt{RP}_t - C_t}_{\text{risk-related}} \qquad (\dagger)$$

Note: If $n = 1$ and $\texttt{CR} = S_t = 0$, then $\log(1/P_t^1) = R_t$ and so $P_t^1 = e^{-R_t}$

Extend the horizon to maturity for n-period bond

1. Average both sides forward over n periods (up to maturity) and take expectations

$$E_t\left[\tfrac{1}{n}\sum_{s=0}^{n-1}\Big(E_{t+s}[\log(P_{t+s+1}^{n-s-1}/P_{t+s}^{n-s})] + \mathtt{CR} + S_{t+s}\Big)\right] = E_t\left[\tfrac{1}{n}\sum_{s=0}^{n-1}\Big(R_{t+s} + \mathtt{RP}_{t+s} - C_{t+s}\Big)\right] \quad (\ddagger)$$

2. After simplifying, (‡) becomes

$$Y_t^n + S_t^n = \underbrace{\frac{1}{n}\sum_{s=0}^{n-1} E_t[R_{t+s}]}_{\text{exp. policy path}} + \underbrace{(\mathtt{RP}_t^n - C_t^n)}_{\text{risk-related}} \quad (\star)$$

   - Yield to maturity: Y_t^n
   - Average expected repo spread: $S_t^n = \frac{1}{n}\sum_{s=0}^{n-1} E_t[S_{t+s}]$
   - Long-term bond risk premium: $\mathtt{RP}_t^n = \frac{1}{n}\sum_{s=0}^{n-1} E_t[\mathtt{RP}_{t+s}]$
   - Long-term bond convexity term: $C_t^n = \frac{1}{n}\sum_{s=0}^{n-1} E_t[C_{t+s}]$

3. **Notes**:
   - Using iterated expectations, "telescoping," and price at maturity (i.e., $\log(P_{t+n}^0) \approx \mathtt{CR}$)

$$E_t\left[\tfrac{1}{n}\sum_{s=0}^{n-1} E_{t+s}[\log(P_{t+s+1}^{n-s-1}/P_{t+s}^{n-s})]\right] \approx \frac{\mathtt{CR}+\log(1/P_t^n)}{n}$$

   - Yield-to-maturity (approximation for coupon bond): $\frac{\mathtt{CR}+\log(1/P_t^n)}{n} + \mathtt{CR} \approx Y_t^n$

Yield to maturity for an n-period bond, Y_t^n

The no-arbitrage condition for default-free long-term bond yields

(move repo spread to the right-hand side)

$$Y_t^n = \underbrace{\frac{1}{n}\sum_{s=0}^{n-1} E_t[R_{t+s}]}_{\text{exp. policy path}} + \underbrace{\overbrace{(\mathtt{RP}_t^n - C_t^n)}^{\text{risk-related}} - S_t^n}_{\text{term premium}} \qquad \text{(bond yield)}$$

- Average expected repo spread: $S_t^n = \frac{1}{n}\sum_{s=0}^{n-1} E_t[S_{t+s}]$
- Risk premium: $\mathtt{RP}_t^n = \frac{1}{n}\sum_{s=0}^{n-1} E_t[\mathtt{RP}_{t+s}]$
- Convexity term: $C_t^n = \frac{1}{n}\sum_{s=0}^{n-1} E_t[C_{t+s}]$

1. **Repo earnings** reduce the yield to maturity
   - They increase the bond price; this is important for **on-the-run bonds**
2. The **risk premium** is (roughly) linear in maturity
   - Time-varying risk premiums produce failure of expectations hypothesis
3. The **convexity** term is (roughly) quadratic in maturity
4. **Policy path** of short-term nominal rates
   - Movements in the expected policy path translate directly into long-term yields
   - Uncertainty about how the short-term interest rate moves gets embedding in the risk premiums for long-term bonds
     - Greater uncertainty about the target rate at the next FOMC meeting produces increased term premiums

The effect of the "measured pace" language

1. At the May 2004 meeting, the FOMC introduced the phrase

   "policy accommodation can be removed at a pace that is likely to be measured"

2. The "measured pace" language came to be interpreted as indicating that the Fed would continue its 25-basis-point-per-meeting increase in the target rate at the following meeting
   - Consequently, it would be at least two meetings before the path would be changed
   - This reduced potential variation in the meeting-to-meeting changes in the target rate
   - The reduction in the variance of the short-term interest rate reduced the risk premiums for long term bond and contributed to the decline yields
     - This effect explains at least of the so-called conumdrum

3. Actions help words speak loudly
   - At first, bond yields responded to the employment reports, but after the Fed continued with the 25-basis-point-per-meeting increase, the response of the yields declined substantially

Market forecasts and the Fed's reaction function

Scenario

1. Suppose the market's forecast of the economy changes in such a way that
   - given the market's understanding of the Fed's reaction function
   - the forecast of the policy path is increased

Consequences

2. The Fed is now expected to set rates higher in the future
   - relative to what was previously expected
3. Long-term yields increase today
4. Market commentary: The Fed is "behind the curve"
   - It didn't react soon enough!
   - It was caught off guard!

The effect of Quantitative Easing

Actions helping words speak loudly (again)

Scenario

1. The Fed expands its balance sheet dramatically
2. The market interprets this undertaking as a signal that the Fed
   - won't raise rates for a substantial period of time
   - will provide ample advance warning as to when it intends to raise rates in the future
3. The expected policy path shifts downward and long-terms yields decline

Introducing the Fisher equation

1. It expresses the relation between
   - nominal interest rate
   - real interest rate
   - expected inflation rate
   - risk-related terms
2. It is the no-arbitrage condition for a default-free bond for which
   - the price is fixed in real terms
   - the dividend rate equals to the real rate of interest

   A **real floater**

3. As a preliminary, consider: The no-arbitrage condition for a default-free bond for which
   - the price is fixed in nominal terms
   - the dividend rate equals the nominal rate of interest

   A **nominal floater**

   The no-arbitrage condition is satisfied by this nominal floater

$$\underbrace{E_t[\log(P_{t+1}/P_t)]}_{=0} + d_t = R_t + \underbrace{\mathtt{RP}_t - C_t}_{=0} \qquad (\dagger)$$

Deriving the Fisher equation: Real floater in nominal terms

1. The nominal price P_t of a real floater is the **price level** (e.g., the CPI_t)
2. The expected growth rate (of the nominal price) is the **expected inflation rate** π_t

$$E_t[\log(P_{t+1}/P_t)] = \pi_t$$

3. The dividend rate is the risk-free **real rate of interest** r_t

$$d_t = \frac{D_t}{P_t} = \frac{r_t \times P_t}{P_t} = r_t$$

4. The no-arbitrage condition delivers the **Fisher equation**

$$\pi_t + r_t = R_t + \underbrace{\mathtt{RP}_t^\pi - C_t^\pi}_{\text{risk-related}} \qquad \text{(Fisher)}$$

- The risk-related terms are expressed in nominal terms
- (Fisher) does not tell us the direction of causality
  - For example: Does π_t cause R_t or does R_t cause π_t?

5. Note: (Fisher) can be reinterpreted as the no-arbitrage condition for a **foreign floater**
   - Uncovered interest parity: where π_t is **expected exchange rate depreciation** and r_t is the **foreign risk-free interest rate**

Applying the Fisher equation: Replace nominal rates

1. Use the Fisher equation to replace nominal rates in the expression for bond yields

$$\pi_t + r_t = R_t + \overbrace{\mathtt{RP}_t^\pi - C_t^\pi}^{\text{risk-related}} \qquad \text{(Fisher)}$$

$$Y_t^n = \underbrace{\frac{1}{n}\sum_{s=0}^{n-1} E_t[R_{t+s}]}_{\text{exp. policy path}} + \underbrace{\overbrace{(\mathtt{RP}_t^n - C_t^n)}^{\text{risk-related}} - S_t^n}_{\text{term premium}} \qquad \text{(bond yield)}$$

2. **The result:**

$$Y_t^n = \overbrace{\underbrace{\frac{1}{n}\sum_{s=0}^{n-1} E_t[r_{t+s}]}_{\text{path of exp. real rate}} + \underbrace{\frac{1}{n}\sum_{s=0}^{n-1} E_t[\pi_{t+s}]}_{\text{path of exp. inflation}}}^{\text{not the exp. policy path}}$$

$$+ \overbrace{\underbrace{(\mathtt{RP}_t^n - C_t^n) - S_t^n}_{\text{term premium}} - \underbrace{\frac{1}{n}\sum_{s=0}^{n-1} E_t[\mathtt{RP}_t^\pi - C_t^\pi]}_{\text{inflation-risk terms}}}^{\text{not the term premium}}$$